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Boundary values as Hamiltonian variables. I. New Poisson brackets

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Abstract

The ordinary Poisson brackets in field theory do not fulfil the Jacobi identity if boundary values are not reasonably fixed by special boundary conditions. We show that these brackets can be modified by adding some surface terms to lift this restriction. The new brackets generalize a canonical bracket considered by Lewis, Marsden, Montgomery and Ratiu for the free boundary problem in hydrodynamics. Our definition of Poisson brackets permits to treat boundary values of a field on equal footing with its internal values and directly estimate the brackets between both surface and volume integrals. This construction is applied to any local form of Poisson brackets. A prescription for δ-function on closed domains and a definition of the *full* variational derivative are proposed.

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1 Introduction

Field theory canonical formalism has some specific features which are absent in mechanics. The need to deal with quantities integrated over some region of space and to integrate by parts leeds in some cases to the appearance of surface integrals in the hamiltonian and(or) in Poisson brackets. Mathematicians usually prefer to consider all these surface integrals to be zero and construct their refined formal variational calculus by identifying integrands which are different by divergencies [1]. But in field theory in some cases these surface terms are not zero and bear physical meaning. So, this paper is devoted to an extension of the hamiltonian formalism onto these divergent terms or informalizing the formal variational calculus.

It seems quite natural to require that for closed systems boundary values should have equal rights with the internal ones and be determined exclusively by initial conditions together with dynamical equations. Here we intend to decline any nondynamical boundary conditions, at least at the first stage of investigation. Afterwards they can be restored as any other constraints which can be put on the initial values of dynamical variables. We believe that such an approach can help us in the following papers of this series to solve some physical problems that are intractable by other methods. So, though the problem treated in this paper is a mathematical one, our motivations are physical.

Let us first remind some important previous results on the problem.

In the famous work by Regge and Teitelboim [2] it was shown that to make the hamiltonian dynamics of the gravitational field in asymptotically flat space existent and nontrivial it was necessary to include into the hamiltonian the surface integrals of special kind. This paper also contains, but not in a very explicit form, the acknowledgement of the physical meaning of surface integrals arising in the evaluation of Poisson brackets, because it was found that they were in correspondence with the surface terms in the hamiltonian through the algebra

$$\{H(N, N^i), H(M, M^j)\} = H(L, L^k),$$

where

$$L = N^i M_{,i} - M^i N_{,i},$$

$$L^k = \gamma^{kl}(N M_{,l} - M N_{,l}) + N^l M^k_{,l} - M^l N^k_{,l},$$

and popular boundary conditions are adopted for canonical variables γ_{ij}, π^{ij} and functions $N(x), M(x), N^i(x), M^j(x)$. In our paper [3] it was shown that this correspondence could be exploited under more general boundary conditions for explicit evaluation of the surface terms in hamiltonian. The essence of the method was the independence of formally defined canonical Poisson brackets

$$\{F, G\} = \int \left[\frac{\delta F}{\delta \gamma_{ij}(x)} \frac{\delta G}{\delta \pi^{ij}(x)} - \frac{\delta G}{\delta \gamma_{ij}(x)} \frac{\delta F}{\delta \pi^{ij}(x)} \right] d^3x,$$

on the surface integrals in the hamiltonians F and G. This property of standard Poisson brackets is in full analogy with the one mentioned in Arnold's book on classical mechanics [4], where the two functions, defined only up to constants, give their Poisson bracket exactly, not up to a constant. So, if in mechanics the kernel of Poisson bracket consists of constants, in field theory the kernel of the ordinary bracket includes surface terms also.

In the papers [5], devoted to the study of the Korteweg-de Vries(KdV) equation, such nonstandard features as the noncommutativity of variational derivatives and violation of the Jacobi identity were observed. In connection with this difficulties different modifications of the Gardner bracket were proposed [5],[6],[7]. Their comparison can be found in Ref. [8]. Unfortunately, these papers were unknown to us during our work on Ashtekar's variables [9], where similar observations were made. Seemingly, such observations were made by mathematicians long before [10],[11]. In our paper [9] it was conjectured that the general criterion for the choice of boundary conditions in the hamiltonian approach to field theory should be the fulfilment of the Jacobi identity for the standard Poisson bracket.

As we recognized from the very readable book by Olver [12], when studying surface waves of the ideal fluid Lewis, Marsden, Montgomery and Ratiu (LMMR) [13] proposed a modified form of canonical Poisson bracket[2]

$$\{F,G\} = \int_\Omega \left[\frac{\delta^\wedge F}{\delta q(x)}\frac{\delta^\wedge G}{\delta p(x)} - \frac{\delta^\wedge G}{\delta q(x)}\frac{\delta^\wedge F}{\delta p(x)}\right] d^n x$$

$$+ \oint_{\partial\Omega}\left[\left.\frac{\delta^\wedge F}{\delta q(x)}\right|_{\partial\Omega}\frac{\delta^\vee G}{\delta p(x)} + \frac{\delta^\vee F}{\delta q(x)}\left.\frac{\delta^\wedge G}{\delta p(x)}\right|_{\partial\Omega}\right] dS$$

$$- \oint_{\partial\Omega}\left[\left.\frac{\delta^\wedge G}{\delta q(x)}\right|_{\partial\Omega}\frac{\delta^\vee F}{\delta p(x)} + \frac{\delta^\vee G}{\delta q(x)}\left.\frac{\delta^\wedge F}{\delta p(x)}\right|_{\partial\Omega}\right] dS,$$

where the two components of variational derivative were defined through the formula:

$$D_q F(q,p)\cdot\delta q = \int_\Omega \frac{\delta^\wedge F}{\delta q}\cdot \delta q d^n x + \oint_{\partial\Omega}\frac{\delta^\vee F}{\delta q}\cdot \delta q|_{\partial\Omega} dS, \tag{1.1}$$

where partial Fréchet derivative was used in the l.h.s., and through the analogous formula for the momentum derivative. This bracket was accompanied by boundary conditions that LMMR considered as necessary

$$\frac{\delta^\vee F}{\delta q}\frac{\delta^\vee G}{\delta p} - \frac{\delta^\vee G}{\delta q}\frac{\delta^\vee F}{\delta p} = 0. \tag{1.2}$$

LMMR mentioned, that earlier the hamiltonian structure for surface waves of the ideal fluid for the potential flow was discovered by Zakharov [14].

Here we will generalize the LMMR formula in such a way that the new bracket will fulfil the Jacobi identity without any boundary conditions. It will permit us to consider on formally equal grounds both volume and surface hamiltonians. The boundary values of hamiltonian variables are now obeying their own hamiltonian equations and fixation of some boundary conditions is simply a new constraint, that should be examined, according to the Dirac procedure, for the presence of secondary and higher constraints. Such a generalization of the LMMR bracket seems necessary also because this bracket of two differentiable functionals can be a functional not differentiable in the sense of (1.1) and (1.2). Therefore, in general, it is not possible even to check the Jacobi identity for the LMMR bracket.

[2]It is interesting to note that the LMMR work was reported at the same conference and published in the same journal issue as the work by Buslaev, Faddeev and Takhtajan [7], in which the Gardner KdV bracket had been modified.

We announce more general new formulae for Poisson brackets and prove for some important cases that they fulfil the new definition of Poisson brackets *without discarding any surface integrals*. The cases where the proofs are demonstrated are 1)ultralocal bracket with constant structure matrix (the canonical case); 2)ultralocal bracket, depending on field variables but not on their derivatives (Lie-Poisson brackets are the most popular examples); 3)nonultralocal brackets with constant structure matrix (Gardner-Zakharov-Faddeev bracket for KdV equation may serve as an example).

Plan of this paper is as follows.

In Sec.2 we introduce necessary notations and give briefly the mathematical background: definitions, lemmas and formulae. Sec.3 contains the primary mathematical motivation for the construction of the new brackets in the ultralocal case: the idea that Poisson bracket should generate the full variation of a local functional. In Sec.4 the method for constructing general local brackets is presented. It is based on integration by parts of the local formula and gives some new proposals about handling distributions in this case. Of course, these calculations can be justified only "a posteriori" in Sec.6. Sec.5 is devoted to a definition of the full variational derivative as a distribution. Here we also present the most unexpected result of the paper: a multiplication rule for derivatives of the characteristic function. This rule permits to write the new Poisson brackets in the form of the old ones but with the new variational derivatives. Sec.6 contains three different proofs of the Jacobi identity for the new brackets. The simplest proof is applicable only for ultralocal brackets with constant structure matrix. The more general proof is applicable to ultralocal brackets dependent on field variables. This proof heavily relays upon Aldersley's results [11] for higher Eulerian operators. The proof of Jacobi identity for nonultralocal brackets with constant coefficients is demonstrated too. We are sure that general proof for arbitrary local brackets along the lines of these partial proofs can also be constructed but it is clear, that it should be rather long. Probably, some other ways for this proof would be shorter. In the Conclusion we give a rather short resume, because a detailed comparison of our approach with Regge-Teitelboim's and LMMR's treatments of surface terms in the Hamiltonian formalism is postponed to next papers of this series. Appendix contains a collection of different ways of presentation of the new brackets.

We plan to discuss the related physical problems of string theory, gauge and gravitation fields in further papers of this series.

2 Notations and mathematical background

In this paper we use the local coordinate language and instead of the manifold with a boundary consider a domain Ω in R^n having a smooth boundary $\partial\Omega$. The characteristic function of this domain is $\theta_\Omega = \theta(P_\Omega)$, where equation $P_\Omega(x^1, ..., x^n) = 0$ defines the boundary. We do not expect that global formulation can meet with serious difficulties.

Definition 2.1 *An integral over a compact domain Ω of a function of field variables $\phi^A(x), A = 1, ..., p$ and their partial derivatives $D_J\phi^A$ up to some finite order*

$$F = \int_\Omega d^n x f(\phi_A(x), D_J\phi_A(x))$$

is called a local functional.

All the functions f and ϕ_A as well as their variations throughout the paper are supposed infinitely smooth, i.e. $C^\infty(R^n)$. We use the multi-index notations $J = (j_1, ..., j_n)$

$$D_J = \frac{\partial^{|J|}}{\partial^{j_1} x^1 ... \partial^{j_n} x^n}, \qquad |J| = j_1 + ... + j_n.$$

Binomial coefficients for multi-indices are

$$\binom{J}{K} = \binom{j_1}{k_1} ... \binom{j_n}{k_n},$$

where ordinary binomial coefficients are

$$\binom{j}{k} = \begin{cases} j!/(k!(j-k)!) & \text{if } 0 \leq k \leq j; \\ 0 & \text{otherwise.} \end{cases}$$

As the number of sums in some formulae of this paper is considerably more than ten, we write only one sign of summing without displaying the indices of summation. According to this half-Einstein rule, sum over all repeated indices should be understood. Only in those cases, where it is not so, we display the summation indices. Also, we do not show the limits of summation, because they are always natural, i.e. outside them the summand is simply zero. Such a nice property of binomial coefficients considerably helps us in many cases of changing of the orders of summation. There is also a temptation to remove the useless $d^n x$ in the integrals and to write the arguments only when they can be mixed. In principle all integrals over finite domains would be better written over all R^n with the help of characteristic function, but we will use in parallel the notions

$$\int_\Omega f \qquad \text{or} \qquad \int \theta_\Omega f \qquad \text{or} \qquad \int \theta(P_\Omega) f.$$

We denote as $\mathcal{A}$ the space of local functionals . It is very important that this space includes functionals with integrands depending on derivatives of arbitrary order [15]. Otherwise the Poisson brackets could go out of $\mathcal{A}$.

Definition 2.2 *A bilinear operation* $\{\cdot, \cdot\}$ *such that for any* $F, G, H \in \mathcal{A}$
1) $\{F, G\} \in \mathcal{A}$*;*
2) $\{F, G\} = -\{G, F\} \qquad mod\ (Div)$*;*
3) $\{\{F, G\}, H\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} = 0 \qquad mod\ (Div)$*;*
is called the standard field theory Poisson bracket.

Definition 2.3 *A bilinear operation* $\{\cdot, \cdot\}$ *such that for any* $F, G, H \in \mathcal{A}$
1) $\{F, G\} \in \mathcal{A}$*;*
2) $\{F, G\} = -\{G, F\}$*;*
3) $\{\{F, G\}, H\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} = 0$*;*
is called the new field theory Poisson bracket.

Definition 2.4[12, Definition 5.70] *Higher eulerian operators* E_A^J *are defined through the formula of full variation of local functional*

$$\delta F = \sum \int_\Omega D_J \Big(E_A^J(f) \delta\phi_A \Big). \tag{2.1}$$

Lemma 2.5[12, Statement 5.72] *Higher eulerian operators can be given by the formula*

$$E_A^J(f) = \sum_K (-1)^{|K|+|J|} \binom{K}{J} D_{K-J} \frac{\partial f}{\partial \phi_A^{(K)}}. \tag{2.2}$$

Usual variational derivative (or Euler-Lagrange derivative) is the eulerian operator of zeroth order. Let us mention that if J is not contained in K, then all quantities having multi-index $(K - J)$ are zero. The sums over J in Eq.(2.1) and over K in Eq.(2.2) are really finite because local functional can depend only on a finite number of derivatives according to Definition 2.1.

Lemma 2.6[12, Statement 5.76] *The eulerian operators have a property*

$$E_A^J(D_I f) = E_A^{J-I}(f).$$

Just this property was the reason for the first appearance of these operators in paper [16].

Lemma 2.7[11, Proposition 3.1] *Eulerian operator of a product of two local functionals is*

$$E_A^K(fg) = \sum_L (-1)^{|K|+|L|} \binom{L}{K} \Big(E_A^L(f) D_{L-K} g + E_A^L(g) D_{L-K} f \Big).$$

Lemma 2.8[11, Theorem 2.1] *Product of eulerian operators is*

$$E_A^I E_B^J(f) = \sum_K (-1)^{|K|} \binom{J+K}{J} E_A^{I-K} \frac{\partial f}{\partial \phi_B^{(J+K)}}.$$

Lemma 2.9[11, Proposition 1.1]

$$\frac{\partial f}{\partial \phi_A^{(J)}} = \sum_K \binom{K}{J} D_{K-J} E_A^K(f).$$

Lemma 2.10[11, Lemma 2.2]

$$\frac{\partial}{\partial \phi_A^{(I)}} D_J f = \sum_K \binom{J}{K} D_{J-K} \frac{\partial f}{\partial \phi_A^{(I-K)}},$$

Let us mention that notations in [11] are different because multi-indices are not used there and because the definition of eulerian operator E_A^I differs by factor $(-1)^{|I|}$.

We also need combinatorial identities

Lemma 2.11[11, Lemma 1.1]

$$\sum_{l=k}^{j} (-1)^l \binom{l}{i} \binom{j-k}{l-k} = (-1)^j \binom{k}{j-i},$$

that can be written as

$$\sum_{l=k}^{j} \frac{(-1)^l l!}{(l-i)!(l-k)!(j-l)!} = (-1)^j \frac{i!k!}{(j-i)!(j-k)!(i+k-j)!}.$$

Lemma 2.12[17, p.616]

$$\sum_{l=0}^{j} \binom{i}{l} \binom{j-k}{j-l} = \binom{i+j-k}{j},$$

that can be also written as

$$\sum_{l=0}^{j} \frac{1}{l!(j-l)!(l-k)!(i-l)!} = \frac{(i+j-k)!}{i!j!(j-k)!(i-k)!}.$$

Definition 2.13[12, Definition 5.28] *The partial Fréchet derivative of a function f is a differential operator D_{f_A}, defined for arbitrary q_A as*

$$D_{f_A}(q) = \frac{d}{d\epsilon} f(\phi_A + \epsilon q_A(\phi)) \bigg|_{\epsilon=0}.$$

In our case

$$D_{f_A} = \sum_I \frac{\partial f}{\partial \phi_A^{(I)}} D_I. \tag{2.3}$$

The Leibnitz rule is

$$D_J(fg) = \sum_K \binom{J}{K} D_K f D_{J-K} g. \tag{2.4}$$

3 Motivation for the new brackets from the full variation formula

As a rule, the Poisson brackets are given by the formula

$$\{F, G\} = \sum \int_\Omega \int_\Omega \frac{\delta F}{\delta \phi_A(x)} \frac{\delta G}{\delta \phi_B(y)} \{\phi_A(x), \phi_B(y)\},$$

where variational derivative is believed to be the zeroth order eulerian operator (Euler-Lagrange derivative)

$$\frac{\delta}{\delta \phi_A} = E_A^0 = \sum (-1)^{|J|} D_J \frac{\partial}{\partial \phi_A^{(J)}}$$

and where we do not care for any surface integrals because all of them are supposed to be zero. Here we limit our attention to ultralocal Poisson brackets. The more general case will be treated in the next Section.

Definition 3.1 *The standard Poisson bracket is called ultralocal if*

$$\{\phi_A(x), \phi_B(y)\} = I_{AB}\delta(x-y),$$

where the so called implectic [18] *operator I_{AB} (or structure matrix) can depend on the field variables $\phi_A(x)$ and their derivatives $D_K \phi_A(x)$.*

These Poisson brackets, together with the local functional H, called hamiltonian, generate a variation of any local functional F under fixed boundary values of $\phi_A, D_J\phi_A$ according to the formula

$$\delta_H F = \{F, H\} = \sum \int_\Omega \frac{\delta F}{\delta \phi_A} \delta_H \phi_A,$$

where

$$\delta_H \phi_A = \sum I_{AB} \frac{\delta H}{\delta \phi_B}. \tag{3.1}$$

The new Poisson brackets we are searching for should analogously generate for a given hamiltonian H *a full variation* of a local functional F in accordance with Eq.(2.1)

$$\delta_H F = \{F, H\} = \sum \int_\Omega D_J \Big(E_A^J(f) \delta_H \phi_A \Big),$$

where variations $\delta_H \phi_A$ of field variables are linearly dependent not only on $E_B^0(h)$, but also on higher eulerian operators (2.2)

$$\delta_H \phi_A = \sum I_{AB}^{(K)} E_B^K(h).$$

Evidently, $I_{AB}^{(0)} = I_{AB}$, and other coefficients $I_{AB}^{(K)}$ will be found below. Really, they are distributions, and this aspect will be treated in the next two Sections. Here we need them "in weak sense", i.e. as functionals defined on the standard smooth functions. We will show, that these coefficients can be found from the requirement of antisymmetry of Poisson brackets, i.e.,

$$\sum \int_\Omega D_J \Big(E_A^J(f) I_{AB}^{(K)} E_B^K(h) \Big) = - \sum \int_\Omega D_J \Big(E_A^J(h) I_{AB}^{(K)} E_B^K(f) \Big).$$

Let us consider this condition perturbatively in the order of eulerian operators $|J| + |K|$. In the zeroth order the antisymmetry is fulfilled due to the related property of the standard bracket

$$I_{AB}^{(0)} = -I_{BA}^{(0)}.$$

In the first order we should have

$$\sum_{A,B} \sum_{|K|=1} \int_\Omega E_A^0(f) I_{AB}^{(K)} E_B^K(h) + \sum_{A,B} \sum_{|J|=1} \int_\Omega D_J \Big(E_A^J(f) I_{AB}^{(0)} E_B^0(h) \Big)$$

$$= - \sum_{A,B} \sum_{|K|=1} \int_\Omega E_A^0(h) I_{AB}^{(K)} E_B^K(f) - \sum_{A,B} \sum_{|J|=1} \int_\Omega D_J \Big(E_A^J(h) I_{AB}^{(0)} E_B^0(f) \Big).$$

If we regroup the terms and exploit the zeroth order antisymmetry, then after relabeling some indices $(A \leftrightarrow B)$, $(J \leftrightarrow K)$ the above relation can be written as

$$\sum_{A,B} \sum_{|K|=1} \int_\Omega \Big(E_A^0(f) I_{AB}^{(K)} E_B^K(h) + E_A^0(h) I_{AB}^{(K)} E_B^K(f) \Big)$$

$$= \sum_{A,B} \sum_{|K|=1} \int_\Omega D_K \Big(E_A^0(f) I_{AB}^{(0)} E_B^K(h) + E_A^0(h) I_{AB}^{(0)} E_B^K(f) \Big).$$

Taking into account the linear independence of eulerian operators we conclude that

$$\sum_{A,B}\sum_{|K|=1}\int_\Omega E^0_A(f)I^{(K)}_{AB}E^K_B(h)=\sum_{A,B}\sum_{|K|=1}\int_\Omega D_K\Big(E^0_A(f)I^{(0)}_{AB}E^K_B(h)\Big). \tag{3.2}$$

Therefore, we succeed in determining the coefficients $I^{(K)}_{AB}$ for the $|K|=1$ case.

Then, let us consider the next order

$$\sum_{A,B}\sum_{|J|=2}\int_\Omega D_J\Big(E^J_A(f)I^{(0)}_{AB}E^0_B(h)\Big)+\sum_{A,B}\sum_{|J|=1}\sum_{|K|=1}\int_\Omega D_J\Big(E^J_A(f)I^{(K)}_{AB}E^K_B(h)\Big)$$
$$+\sum_{A,B}\sum_{|K|=2}\int_\Omega E^0_A(f)I^{(K)}_{AB}E^K_B(h)=-\sum_{A,B}\sum_{|J|=2}\int_\Omega D_J\Big(E^J_A(h)I^{(0)}_{AB}E^0_B(f)\Big)$$
$$-\sum_{A,B}\sum_{|J|=1}\sum_{|K|=1}\int_\Omega D_J\Big(E^J_A(h)I^{(K)}_{AB}E^K_B(f)\Big)-\sum_{A,B}\sum_{|K|=2}\int_\Omega E^0_A(h)I^{(K)}_{AB}E^K_B(f). \tag{3.3}$$

If we take into account the result obtained before (3.2), then the second terms in the l.h.s. and in the r.h.s. of equation (3.3) are mutually cancelled. Making the same procedure as was used for the first order we find

$$\sum_{|K|=2}\int_\Omega E^0_A(f)I^{(K)}_{AB}E^K_B(h)=\sum_{|K|=2}\int_\Omega D_K\Big(E^0_A(f)I^{(0)}_{AB}E^K_B(h)\Big).$$

So, it is clear that from the only requirement of antisymmetry we, step by step, become convinced that the Poisson bracket should be written as

$$\{F,H\}=\sum\int_\Omega D_{J+K}\Big(E^J_A(f)I_{AB}E^K_B(h)\Big). \tag{3.4}$$

Now we are able to formulate

Theorem 3.2 *Formula* (3.4) *gives a new Poisson bracket, if its zeroth order* ($|J|=0=|K|$) *term is a standard ultralocal Poisson bracket and the structure coefficients do not depend on the derivatives of the field variables.*

Proof. The antisymmetry is clear from the construction. Evidently, the bracket is a local functional and all that we are to prove is the Jacobi identity. This proof will be given in Sec.6, but first there we will give a considerably more simple proof for the case when I_{AB} are constants.

Remark. It is not difficult to include the case, when I_{AB} also depends on field derivatives, but it makes the proof of Jacobi identity even longer and the conditions for standard brackets are not so transparent.

4 Surface terms and distributions

The standard field theory Poisson bracket [12]

$$\{F,G\}=\sum\int_\Omega\int_\Omega E^0_A(f(x))E^0_B(g(y))\{\phi_A(x),\phi_B(y)\}$$

is a special case, which is true only under assumption that all surface terms arising when integrating by parts are zero, of the formula

$$\{F,G\} = \sum \int_\Omega \int_\Omega \frac{\partial f}{\partial \phi_A^{(J)}(x)} \frac{\partial g}{\partial \phi_B^{(K)}(y)} \{D_J^{(x)}\phi_A(x), D_K^{(y)}\phi_B(y)\}, \tag{4.1}$$

or

$$\{F,G\} = \sum \int_\Omega \int_\Omega D_{f_{A(x)}} D_{g_{B(y)}} \{\phi_A(x), \phi_B(y)\},$$

where Fréchet derivatives (2.3) are used.

Definition 4.1 *The standard field theory Poisson bracket is called local if*

$$\{\phi_A(x), \phi_B(y)\} = \frac{1}{2}\sum_L \Big(I_{AB}^L(x) D_L^{(x)} - I_{BA}^L(y) D_L^{(y)} \Big)\delta(x-y), \tag{4.2}$$

where the sum is of finite range in $|L|$.

Usually, derivatives over only one of the two arguments are present in the formulae like (4.2), because of the widely used relations

$$\Big(D_J^{(x)} - (-1)^{|J|} D_J^{(y)} \Big)\delta(x-y) = 0, \tag{4.3}$$

also accompanied by

$$I_{AB}^L = (-1)^{|L|+1} I_{BA}^L.$$

But if not all surface terms, arising in integration by parts, are zero, then (4.3) are not true. This observation was made by the author in [9] where it had been realized that the problem could be reduced to the definition of integrals like

$$\int_\Omega \int_\Omega f(x) g(y) D_J^{(x)} D_K^{(y)} \delta(x-y), \tag{4.4}$$

for finite domain when the test functions were nonzero on its boundary.

The theory of distributions [19] considers them as defined on the open space domains. In the literature known to us related problems of defining distributions on closed domains were discussed in books [20],[21], but the unique answer how to define the integral in (4.4) is absent there. So, here we propose a Rule which is in accordance with the results of the previous Section.

Rule 4.2

$$\int_\Omega \int_\Omega f(x) g(y) D_J^{(x)} D_K^{(y)} \delta(x-y) = \int_\Omega D_K f D_J g. \tag{4.5}$$

This Rule is different from that proposed in [9], because the rules are compatible with the different Poisson brackets.

Taken together, Eqs.(4.1) and (4.5) give us a possibility to obtain not only the previously found expression (3.4) for ultralocal brackets, but also a more general result. Let us substitute (4.2) into (4.1)

$$\{F,G\} = \frac{1}{2}\sum \int_\Omega \int_\Omega \frac{\partial f}{\partial \phi_A^{(J)}(x)} \frac{\partial g}{\partial \phi_B^{(K)}(y)}$$

$$\times D^{(x)}_J D^{(y)}_K \Big(\Big(I^L_{AB}(x) D^{(x)}_L - I^L_{BA}(y) D^{(y)}_L \Big) \delta(x-y) \Big),$$

and exploit the Leibnitz rule (2.4)

$$\{F,G\} = \frac{1}{2} \sum \int_\Omega \int_\Omega \frac{\partial f}{\partial \phi^{(J)}_A(x)} \frac{\partial g}{\partial \phi^{(K)}_B(y)}$$

$$\times \Big(\binom{J}{M} D^{(x)}_M I^L_{AB}(x) D^{(x)}_{L+J-M} D^{(y)}_K - \binom{K}{M} D^{(y)}_M I^L_{BA}(y) D^{(x)}_J D^{(y)}_{L+K-M} \Big) \delta(x-y).$$

Then take off one of the integrations by Rule 4.2

$$\{F,G\} = \frac{1}{2} \sum \int_\Omega \Big(\binom{J}{M} D_{L+J-M} \frac{\partial g}{\partial \phi^{(K)}_B} D_K \Big(\frac{\partial f}{\partial \phi^{(J)}_A} D_M I^L_{AB} \Big)$$

$$- \binom{K}{M} D_{L+K-M} \frac{\partial f}{\partial \phi^{(J)}_A} D_J \Big(\frac{\partial g}{\partial \phi^{(K)}_B} D_M I^L_{BA} \Big) \Big).$$

Once more using the Leibnitz rule

$$\{F,G\} = \frac{1}{2} \sum \int_\Omega \Big(\binom{J}{M} \binom{K}{N} D_{L+J-M} \frac{\partial g}{\partial \phi^{(K)}_B} D_{N+M} I^L_{AB} D_{K-N} \frac{\partial f}{\partial \phi^{(J)}_A}$$

$$- \binom{K}{M} \binom{J}{N} D_{L+K-M} \frac{\partial f}{\partial \phi^{(J)}_A} D_{N+M} I^L_{BA} D_{J-N} \frac{\partial g}{\partial \phi^{(K)}_B} \Big),$$

and making changes ($J \leftrightarrow K$), ($A \leftrightarrow B$) in the second term we obtain

$$\{F,G\} = \frac{1}{2} \sum \int_\Omega \binom{J}{M} \binom{K}{N} D_{N+M} I^L_{AB}$$

$$\times \Big(D_{K-N} \frac{\partial f}{\partial \phi^{(J)}_A} D_{L+J-M} \frac{\partial g}{\partial \phi^{(K)}_B} - D_{K-N} \frac{\partial g}{\partial \phi^{(J)}_A} D_{L+J-M} \frac{\partial f}{\partial \phi^{(K)}_B} \Big).$$

Transform here the partial derivatives into eulerian operators according to Lemma 2.9

$$\{F,G\} = \frac{1}{2} \sum \int_\Omega \binom{J}{M} \binom{K}{N} \binom{P}{J} \binom{Q}{K} D_{N+M} I^L_{AB}$$

$$\times \Big(D_{P-J+K-N} E^P_A(f) D_{Q-K+L+J-M} E^Q_B(g) - (F \leftrightarrow G) \Big),$$

make a change $J \to J+K$ and estimate the sum over K according to Lemma 2.12

$$\sum_K \binom{J+K}{M} \binom{P}{J+K} \binom{K}{N} \binom{Q}{K} = \binom{P}{M} \binom{Q}{N} \binom{P+Q-M-N}{P-J-N}.$$

Then we get

$$\{F,G\} = \frac{1}{2} \sum \int_\Omega \binom{P}{M} \binom{Q}{N} \binom{P+Q-M-N}{P-J-N}$$

$$\times D_{N+M} I^L_{AB}\Big(D_{P-J-N} E^P_A(f) D_{Q+L+J-M} E^Q_B(g) - (F \leftrightarrow G)\Big).$$

It is not difficult to get convinced with the help of Leibnitz rule that the obtained result coincides with

$$\frac{1}{2}\sum\int_\Omega D_{P+Q}\Big(E^P_A(f) I^L_{AB} D_L E^Q_B(g) - (F \leftrightarrow G)\Big). \tag{4.6}$$

For ultralocal case $I^L_{AB} = \delta_{L0} I_{AB}$, $I_{AB} = -I_{BA}$, and, evidently, Eq.(3.4) can be reproduced. For a more general case we have

Theorem 4.3 *The new Poisson brackets, corresponding to the standard local Poisson brackets with constant structure matrix, are given by formula* (4.6)*.*

Proof. The antisymmetry is evident, it is also evident that (4.6) gives local functional. The Jacobi identity for this case will be proved in Sec.6.3.

Remark. Evidently, the construction does not restrict us to the case $I^L_{AB} = const$. But the proof of Jacobi identity becomes more difficult in the general case.

5 A full variational derivative and a multiplication rule: towards informal variational calculus

Let us present the standard variational, or Euler-Lagrange, derivative in the form

$$\frac{\delta F}{\delta\phi_A(x)} = E^0_A(f)\theta_\Omega.$$

Then it gives us a full variation

$$\delta F = \sum\int \frac{\delta F}{\delta\phi_A}\delta\phi_A,$$

of a local functional

$$F = \int \theta_\Omega f(\phi_A, D_J\phi_A),$$

only if all surface integrals in the general formula

$$\delta F = \sum\int \theta_\Omega D_J\Big(E^J_A(f)\delta\phi_A\Big),$$

are zero.

Definition 5.1 *A distribution $\delta F/\delta\phi_A$ such that in general case, i.e. for arbitrary smooth variations $\delta\phi_A(x)$,*

$$\delta F = \sum\int \frac{\delta F}{\delta\phi_A}\delta\phi_A, \tag{5.1}$$

will be called the full variational derivative of a local functional F.

Statement 5.2 *The full variational derivative can be written in the form*

$$\frac{\delta F}{\delta\phi_A} = \sum(-1)^{|J|} E^J_A(f) D_J\theta_\Omega, \tag{5.2}$$

where θ_Ω is a characteristic function of the domain of integration Ω.

Proof. Through integration by parts

$$\sum \int (-1)^{|J|} E_A^J(f) D_J \theta_\Omega \delta\phi_A = \sum \int \theta_\Omega D_J \Big(E_A^J(f) \delta\phi_A \Big)$$

$$= \sum \int_\Omega D_J \Big(E_A^J(f) \delta\phi_A \Big).$$

Statement 5.3 *By using* Definition 5.1 *the new Poisson brackets* (4.6) *can be written in the form*

$$\{F, G\} = \sum \int \int \frac{\delta F}{\delta\phi_A(x)} \{\phi_A(x), \phi_B(y)\} \frac{\delta G}{\delta\phi_B(y)}, \tag{5.3}$$

if we admit the following multiplication rule:

Rule 5.4

$$D_J \theta(P_\Omega) \times D_K \theta(P_\Omega) = D_{J+K} \theta(P_\Omega).$$

Remark. Of course, this Rule has its domain of applicability only inside our procedure of calculating the Poisson brackets. Maybe, it can also find its place in the new theory of generalized functions [22].

Proof of the Statement. Let us substitute (5.2) and (4.2) into (5.3) and take off the derivatives from the δ-function through integration by parts

$$\frac{1}{2} \sum (-1)^{|J|+|K|+|L|} \int \int \Big[D_L \Big(D_J (\theta(P_\Omega(x)) E_A^J(f) I_{AB}^L(x) \Big) D_K \theta(P_\Omega(y)) E_B^K(g)$$

$$- D_L \Big(D_K \theta(P_\Omega(y)) E_B^K(g) I_{BA}^L(y) \Big) D_J \theta(P_\Omega(x)) E_A^J(f) \Big] \delta(x - y).$$

Then take off one of integrations with the help of δ-function, and afterwards use the Leibnitz rule

$$\frac{1}{2} \sum (-1)^{|J|+|K|+|L|} \binom{L}{M} \int \Big[D_{J+M} \theta_\Omega D_K \theta_\Omega E_B^K(g) D_{L-M} \Big(E_A^J(f) I_{AB}^L \Big)$$

$$- D_{K+M} \theta_\Omega D_J \theta_\Omega E_A^J(f) D_{L-M} \Big(E_B^K(g) I_{BA}^L \Big) \Big].$$

After exploiting Rule 5.4 and integrating by parts we have

$$\frac{1}{2} \sum (-1)^{|L|+|M|} \binom{L}{M} \int_\Omega D_{J+K+M} \Big[E_B^K(g) D_{L-M} \Big(E_A^J(f) I_{AB}^L \Big)$$

$$- E_A^J(f) D_{L-M} \Big(E_B^K(g) I_{BA}^L \Big) \Big],$$

and after using the Leibnitz rule

$$\frac{1}{2} \sum (-1)^{|L|+|M|} \binom{L}{M} \binom{M}{N} \int_\Omega D_{J+K} \Big[D_N E_B^K(g) D_{L-N} \Big(E_A^J(f) I_{AB}^L \Big)$$

$$-D_N E^J_A(f) D_{L-N}\Big(E^K_B(g) I^L_{BA}\Big)\Big].$$

Summing over M

$$\sum_M (-1)^{|M|}\binom{L}{M}\binom{M}{N} = (-1)^{|L|}\delta_{L,N}, \tag{5.4}$$

completes the proof by giving Eq.(4.6).

Statement 5.5 Rule 4.2 *is a corollary of* Rule 5.4.

Proof. With the help of characteristic function θ_Ω we can write the l.h.s. of (4.5) in the form of integrals over infinite space R^n

$$\int\!\!\int \theta(P_\Omega(x)) f(x) \theta(P_\Omega(y)) g(y) D^{(x)}_J D^{(y)}_K \delta(x-y),$$

then no surface terms arise after integration by parts and we have

$$(-1)^{|J|+|K|}\int\!\!\int D^{(x)}_J\Big(\theta(P_\Omega(x)) f(x)\Big) D^{(y)}_K\Big(\theta(P_\Omega(y)) g(y)\Big)\delta(x-y).$$

Remove one of the two integrations with the help of the δ -function and obtain

$$(-1)^{|J|+|K|}\int D_J(\theta_\Omega f) D_K(\theta_\Omega g),$$

then use the Leibnitz rule

$$(-1)^{|J|+|K|}\sum_{L,M}\binom{J}{L}\binom{K}{M}\int D_L\theta_\Omega D_M\theta_\Omega D_{J-L} f D_{K-M} g,$$

and Rule 5.4. After one more integration by parts over R^n we obtain an integral over Ω

$$\sum_{L,M}(-1)^{|J|+|K|+|L|+|M|}\binom{J}{L}\binom{K}{M}\int_\Omega D_{L+M}\Big(D_{J-L} f D_{K-M} g\Big).$$

Again exploiting the Leibnitz rule and, afterwards, calculating the sum over M,

$$\sum_M (-1)^{|M|}\binom{K}{M}\binom{L+M}{N} = (-1)^{|K|}\binom{L}{N-K},$$

we obtain

$$\sum_{L,N}(-1)^{|J|+|L|}\binom{J}{L}\binom{L}{N-K}\int_\Omega D_{N+J-L} f D_{K+L-N} g.$$

After making a change $N \to N+L$ and calculating the sum over L according to Lemma 2.11 we get the r.h.s. of (4.5). The proof is completed.

6 Proofs of Jacobi identity

6.1 A toy proof

Statement 6.1.1 *For constant structure matrix the Poisson bracket* (3.4) *can be written in the form*

$$\{F,G\} = \sum \int_\Omega Tr(D_{f_A} I_{AB} D_{g_B}), \tag{6.1}$$

where D_{f_A} *is Fréchet derivative* (2.3)*, and*

$$Tr(D_{f_A} I_{AB} D_{g_B}) = \sum I_{AB} D_J \frac{\partial f}{\partial \phi_A^{(I)}} D_I \frac{\partial g}{\partial \phi_B^{(J)}}.$$

Proof. Let us use the Leibnitz rule in Eq.(3.4)

$$\{F,G\} = \sum I_{AB} \int_\Omega \binom{I+J}{M} D_M E_A^I(f) D_{I+J-M} E_B^J(g).$$

By exploiting Lemma 2.5 it can be transformed to

$$\sum I_{AB} (-1)^{|I|+|K|+|J|+|L|} \binom{I+J}{M} \binom{K}{I} \binom{L}{J} \int_\Omega D_{M+K-I} \frac{\partial f}{\partial \phi_A^{(K)}} D_{I-M+L} \frac{\partial g}{\partial \phi_B^{(L)}}.$$

Then by changing the indices $M \to M + I$ and the order of summation with the help of Lemmas 2.11 and 2.12 we are able to estimate the sum

$$\sum_{I,J} (-1)^{|I|+|J|} \binom{I+J}{I+M} \binom{K}{I} \binom{L}{J} = (-1)^{|K|+|L|} \delta_{M,L-K}.$$

As a result we obtain (6.1).

Statement 6.1.2 *The Poisson bracket, given by formula* (3.4)*, fulfils the Jacobi identity when* $I_{AB} = const$.

Proof. Let us transform the expression

$$\{\{F,G\},H\} = \sum I_{AB} I_{CD} \int_\Omega D_I \frac{\partial h}{\partial \phi_B^{(J)}}$$

$$\times D_J \left(\frac{\partial}{\partial \phi_A^{(I)}} (D_K \frac{\partial f}{\partial \phi_C^{(L)}}) D_L \frac{\partial g}{\partial \phi_D^{(K)}} + D_L \frac{\partial f}{\partial \phi_C^{(K)}} \frac{\partial}{\partial \phi_A^{(I)}} (D_K \frac{\partial g}{\partial \phi_D^{(L)}}) \right),$$

with Lemma 2.10, Leibnitz rule and antisymmetry over $C \leftrightarrow D$

$$\{\{F,G\},H\} = \sum \binom{K}{M} \binom{J}{N} I_{AB} I_{CD}$$

$$\times \int_\Omega \Bigg(D_{K+N-M} \frac{\partial^2 f}{\partial \phi_A^{(I-M)} \partial \phi_C^{(L)}} D_I \frac{\partial h}{\partial \phi_B^{(J)}} D_{J+L-N} \frac{\partial g}{\partial \phi_D^{(K)}}$$

$$-D_{K+N-M}\frac{\partial^2 g}{\partial\phi_A^{(I-M)}\partial\phi_C^{(L)}}D_I\frac{\partial h}{\partial\phi_B^{(J)}}D_{J+L-N}\frac{\partial f}{\partial\phi_D^{(K)}}\Bigg).$$

After cyclic permutation we have

$$\{\{F,G\},H\}+\{\{H,F\},G\}+\{\{G,H\},F\}$$

$$=\sum\binom{K}{M}\binom{J}{N}I_{AB}I_{CD}\int_\Omega D_{K+N-M}\frac{\partial^2 f}{\partial\phi_A^{(I-M)}\partial\phi_C^{(L)}}$$

$$\times\left(D_{J+L-N}\frac{\partial g}{\partial\phi_D^{(K)}}D_I\frac{\partial h}{\partial\phi_B^{(J)}}-(g\leftrightarrow h)\right)+\ldots,$$

where dots mean analogous terms with cyclically permuted f,g,h. By changing indices $N\to N+J$, $I\to I+M$, we get

$$\{\{F,G\},H\}+\{\{H,F\},G\}+\{\{G,H\},F\}=\sum\binom{K}{K-M}\binom{J}{J+N}I_{AB}I_{CD}$$

$$\times\int_\Omega D_{J+K+N-M}\frac{\partial^2 f}{\partial\phi_A^{(I)}\partial\phi_C^{(L)}}\left[D_{L-N}\frac{\partial g}{\partial\phi_D^{(K)}}D_{I+M}\frac{\partial h}{\partial\phi_B^{(J)}}-(g\leftrightarrow h)\right]+\ldots$$

So, if we simultaneously change $I\leftrightarrow L$, $J\leftrightarrow K$, $M\leftrightarrow -N$, $A\leftrightarrow C$ and $B\leftrightarrow D$, then the expression in square brackets changes its sign while the coefficient before the brackets stands as itself. Therefore, the sum equals zero, and the Jacobi identity is fulfilled in this simplest case.

6.2 Proof for ultralocal case

Statement 6.2.1 *Ultralocal Poisson brackets, given by formula* (3.4)*, with the coefficients, depending on field variables but not on their derivatives, exactly satisfy the Jacobi identity, if the corresponding standard brackets satisfy it up to total divergences.*

Proof. Let us transform the expression

$$\{\{F,G\},H\}=\sum\int_\Omega D_{I+J}\Big(E_A^I\Big(D_{K+L}(E_C^K(f)I_{CD}E_D^L(g))\Big)I_{AB}E_B^J(h)\Big)$$

according to Lemmas 2.6 and 2.7. Then, taking into account that

$$E_A^M(I_{CD})=\delta_{M0}\frac{\partial I_{CD}}{\partial\phi_A},$$

we obtain

$$\sum(-1)^{|I|+|K|+|L|+|M|}\binom{M}{I-K-L}\int_\Omega D_{I+J}\Big(E_B^J(h)I_{AB}\Big[\delta_{M0}\frac{\partial I_{CD}}{\partial\phi_A}$$

$$\times D_{M+K+L-I}\Big(E_C^K(f)E_D^L(g)\Big)+D_{M+K+L-I}I_{CD}E_A^M\Big(E_C^K(f)E_D^L(g)\Big)\Big]\Big).\tag{6.2}$$

Let us consider the first term in square brackets. As $M = 0$, then the binomial coefficient is not zero only when $I = K + L$, therefore this term becomes

$$\sum \int_\Omega D_{J+K+L}\biggl(I_{AB}\frac{\partial I_{CD}}{\partial \phi_A}E_C^K(f)E_D^L(g)E_B^J(h)\biggr). \tag{6.3}$$

After cyclic permutation of F, G, H and having in mind the symmetry in J, K, L we see that this term gives no impact on the r.h.s. of Jacobi identity if

$$I_{AB}\frac{\partial I_{CD}}{\partial \phi_A} + I_{AD}\frac{\partial I_{BC}}{\partial \phi_A} + I_{AC}\frac{\partial I_{DB}}{\partial \phi_A} = 0. \tag{6.4}$$

But just this condition is necessary [12] for fulfilment of the Jacobi identity by standard Poisson brackets in this case. Therefore, we can now care only for the second term in (6.2).

Once more exploit Lemma 2.7, then the term of interest is equal to

$$\sum(-1)^{|I|+|K|+|L|+|N|}\binom{M}{I-K-L}\binom{N}{M}\int_\Omega D_{I+J}\biggl(I_{AB}E_B^J(h)$$

$$\times D_{M+K+L-I}I_{CD}\Bigl[E_A^N E_C^K(f)D_{N-M}E_D^L(g) + E_A^N E_D^L(g)D_{N-M}E_C^K(f)\Bigr]\biggr).$$

If we take into account antisymmetry of the coefficient before the square bracket under $C \leftrightarrow D$ and its symmetry under $K \leftrightarrow L$, we obtain

$$\sum(-1)^{|I|+|K|+|L|+|N|}\binom{M}{I-K-L}\binom{N}{M}\int_\Omega D_{I+J}\biggl(I_{AB}D_{M+K+L-I}I_{CD}$$

$$\times E_B^J(h)\Bigl[E_A^N E_C^K(f)D_{N-M}E_D^L(g) - E_A^N E_C^K(g)D_{N-M}E_D^L(f)\Bigr]\biggr).$$

After cyclic permutation of F, G, H this expression can be written as

$$\sum(-1)^{|I|+|K|+|L|+|N|}\binom{M}{I-K-L}\binom{N}{M}\int_\Omega D_{I+J}\biggl(E_A^N E_C^K(f)$$

$$\times I_{AB}D_{M+K+L-I}I_{CD}\Bigl[E_B^J(h)D_{N-M}E_D^L(g) - (G \leftrightarrow H)\Bigr]\biggr) + \dots .$$

Let us exploit the Leibnitz rule and get

$$\sum(-1)^{|I|+|K|+|L|+|N|}\binom{M}{I-K-L}\binom{N}{M}\binom{I+J}{P}$$

$$\times\binom{I+J-P}{Q}\binom{I+J-P-Q}{R}\binom{I+J-P-Q-R}{S}$$

$$\times\int_\Omega D_P I_{AB}D_{Q+M+K+L-I}I_{CD}D_R E_A^N E_C^K(f)$$

$$\times\Big[D_{S+N-M}E^L_D(g)D_{I+J-P-Q-R-S}E^J_B(h)-(G\leftrightarrow H))\Big]+\dots.$$

Transform the coefficient before the square bracket according to Lemmas 2.8 and 2.5, i.e., make a substitution

$$D_R E^N_A E^K_C(f)=\sum_{T,U}(-1)^{|U|+|N|}\binom{K+T}{K}\binom{U}{N-T}D_{R+T+U-N}\frac{\partial^2 f}{\partial\phi^{(U)}_A\partial\phi^{(K+T)}_C}.$$

After that, it is possible to simplify the expression before the square brackets by changing indices, order of summation and explicit calculation of the four sums of binomial coefficients.

First, make the changes $T\to T-K$, $M\to M-K$, $N\to N-K$ and estimate the sum over K according to Lemma 2.11

$$\sum_K(-1)^{|K|}\binom{T}{K}\binom{M-K}{I-K-L}\binom{N-K}{M-K}=\binom{N-T}{I-L}\binom{L+N-I}{N-M},$$

(during its calculation a trivial shift of argument is exploited, later we will not mention these details).

Then make redefinitions $Q\to Q+I-M-L$, $S\to S+M-N$, $R\to R+N$ and calculate the sum over M by exploiting Lemma 2.12

$$\sum_M\binom{I+J-P}{I+Q-M-L}\binom{L+N-I}{N-M}\binom{J+M+L-P-Q-R-N}{S+M-N}$$
$$\times\binom{J+M+L-P-Q}{R+N}=\binom{Q+S}{Q}\binom{I+J-P}{N+R}\binom{I+J-P-N-R}{I+Q+S-N-L}.$$

After a new replacement $R\to J+L-P-Q-R-S$ we can estimate the sum over N

$$\sum_N\binom{I+J-P}{N+J+L-P-Q-R-S}\binom{I-N-L+Q+S-R}{I+Q+S-N-L}$$
$$\times\binom{U}{N-T}\binom{N-T}{I-L}=\binom{J+L+U-P-R}{Q+S-T}\binom{U}{I-L}\binom{I+J-P}{R}.$$

And the last summation is

$$\sum_I(-1)^{|I|}\binom{U}{I-L}\binom{I+J-P}{R}\binom{I+J}{P}=(-1)^{|U|+|L|}\binom{P+R}{P}\binom{L+J}{P+R-U}$$

As a result, the term under study becomes

$$\sum\binom{Q+S}{Q}\binom{P+R}{P}\binom{L+J}{P+R-U}\binom{L+J+U-P-R}{Q+S-T}\int_\Omega D_P I_{AB}$$
$$\times D_Q I_{CD}D_{J+L+T+U-P-Q-R-S}\frac{\partial^2 f}{\partial\phi^{(U)}_A\partial\phi^{(T)}_C}\Big[D_R E^J_B(h)D_S E^L_D(g)-(H\leftrightarrow G)\Big].$$

It is not difficult to see that under the simultaneous change $A\leftrightarrow C$, $B\leftrightarrow D$, $J\leftrightarrow L$, $R\leftrightarrow S$, $U\leftrightarrow T$, $P\leftrightarrow Q$ the square bracket changes its sign whereas the coefficient before it does not, i.e. all the expression equals zero. With the previous study (6.3), (6.4) in mind the proof is completed.

6.3 Proof for nonultralocal case

Statement 6.3.1 *Nonultralocal Poisson brackets given by formula* (4.6), *exactly satisfy the Jacobi identity, when* $I^K_{AB} = const$.

Proof. By Lemma 2.6 and changes $I \leftrightarrow J$, $A \leftrightarrow B$ we have

$$\{\{F,G\},H\} = \frac{1}{4}\sum I^N_{CD}\int_\Omega D_{I+J}\Big(\Big(I^K_{AB}D_K E^J_B(h) - E^J_B(h)I^K_{BA}D_K\Big)$$

$$\times E^{I-L-M}_A\Big(E^L_C(f)D_N E^M_D(g) - (F \leftrightarrow G)\Big)\Big),$$

and by using Lemma 2.7 obtain

$$E^{I-L-M}_A\Big(E^L_C(f)D_N E^M_D(g)\Big) = \sum_P (-1)^{|P|+|I|+|L|+|M|}\binom{P}{I-L-M}$$

$$\times\Big(E^P_A E^L_C(f)D_{P-I+L+M+N}E^M_D(g) + E^{P-N}_A E^M_D(g)D_{P-I+L+M}E^L_C(f)\Big).$$

Then let us exploit the symmetry $L \leftrightarrow M$ and change $C \leftrightarrow D$

$$\{\{F,G\},H\} = \frac{1}{4}\sum(-1)^{|P|+|I|+|L|+|M|}\binom{P}{I-L-M}$$

$$\times\int_\Omega D_{I+J}\Big(\Big(I^K_{AB}D_K E^J_B(h) - E^J_B(h)I^K_{BA}D_K\Big)\Big(I^N_{CD}E^P_A E^L_C(f)$$

$$\times D_{L+M+P-I+N}E^M_D(g) - I^N_{DC}E^{P-N}_A E^L_C(g)D_{L+M+P-I}E^M_D(f)\Big)\Big).$$

Make a change $P \to P + N$ in the second term

$$\frac{1}{4}\sum\left[\binom{P}{I-L-M}I^N_{CD} - (-1)^{|N|}\binom{P+N}{I-L-M}I^N_{DC}\right]$$

$$\times(-1)^{|P|+|I|+|L|+|M|}\int_\Omega D_{I+J}\Big(\Big(I^K_{AB}D_K E^J_B(h) - E^J_B(h)I^K_{BA}D_K\Big)$$

$$\times\Big(E^P_A E^L_C(f)D_{M+L+P+N-I}E^M_D(g) - (F \leftrightarrow G)\Big)\Big).$$

Then calculate D_K according to the Leibnitz rule

$$\sum_Q\binom{K}{Q}\Big(D_{K-Q}E^P_A E^L_C(f)D_{Q+M+L+P+N-I}E^M_D(g) - (F \leftrightarrow G)\Big)$$

and, analogously, D_{I+J}

$$\{\{F,G\},H\} = \frac{1}{4}\sum\left[\binom{P}{I-L-M}I^N_{CD} - (-1)^{|N|}I^N_{DC}\binom{P+N}{I-L-M}\right]$$

$$\times(-1)^{|P|+|I|+|L|+|M|}\binom{I+J}{R}\binom{I+J-R}{S}\int_\Omega\Big[I^K_{AB}D_{R+K}E^J_B(h)$$

$$\times\Big(D_S E^P_A E^L_C(f)D_{J-R-S+M+L+P+N}E^M_D(g)-(F\leftrightarrow G)\Big)-I^K_{BA}D_R E^J_B(h)$$

$$\times\sum_Q\binom{K}{Q}\Big(D_{S+K-Q}E^P_A E^L_C(f)D_{J-R-S+M+L+N+P+Q}E^M_D(g)-(F\leftrightarrow G)\Big)\Big]$$

Now we are able to sum over I according to Lemma 2.11

$$\sum_I(-1)^{|I|}\binom{P}{I-L-M}\binom{I+J}{R}\binom{I+J-R}{S}$$

$$=(-1)^{|P|+|L|+|M|}\binom{R+S}{R}\binom{J+L+M}{R+S-P},$$

$$\sum_I(-1)^{|I|}\binom{P+N}{I-L-M}\binom{I+J}{R}\binom{I+J-R}{S}$$

$$=(-1)^{|P|+|L|+|M|+|N|}\binom{R+S}{R}\binom{J+L+M}{R+S-P-N},$$

and obtain

$$\{\{F,G\},H\}=\frac{1}{4}\sum\left[\binom{J+L+M}{R+S-P}I^N_{CD}-I^N_{DC}\binom{J+L+M}{R+S-P-N}\right]\binom{R+S}{R}$$

$$\times\int_\Omega\Big(I^K_{AB}D_{R+K}E^J_B(h)\Big(D_S E^P_A E^L_C(f)D_{J-R-S+M+L+P+N}E^M_D(g)-(F\leftrightarrow G)\Big)$$

$$-I^K_{BA}D_R E^J_B(h)\sum_Q\binom{K}{Q}$$

$$\times\Big(D_{J-R-S+M+L+N+P+Q}E^M_D(g)D_{S+K-Q}E^P_A E^L_C(f)-(F\leftrightarrow G)\Big)\Big).$$

If changes $R\to R-K$ are made in the first term

$$\frac{1}{4}\sum\left[\binom{J+L+M}{R+S-P-K}I^N_{CD}-I^N_{DC}\binom{J+L+M}{R+S-K-P-N}\right]\binom{R+S-K}{R-K}$$

$$\times\int_\Omega I^K_{AB}D_R E^J_B(h)\Big(D_S E^P_A E^L_C(f)D_{J-R-S+M+L+P+N+K}E^M_D(g)-(F\leftrightarrow G)\Big),$$

and $S\to S-K+Q$ in the second

$$-\frac{1}{4}\sum\left[\binom{J+L+M}{R+S+Q-P-K}I^N_{CD}-I^N_{DC}\binom{J+L+M}{R+S+Q-K-P-N}\right]$$

$$\times\binom{R+S+Q-K}{R}\binom{K}{Q}\int_\Omega I^K_{BA}D_R E^J_B(h)$$

$$\times\Big(D_S E_A^P E_C^L(f) D_{J-R-S+M+L+N+P+K} E_D^M(g) - (F \leftrightarrow G)\Big),$$

we obtain

$$\{\{F,G\},H\} = \frac{1}{4}\sum\Bigg[\binom{J+L+M}{R+S-P-K}\binom{R+S-K}{R-K} I_{CD}^N I_{AB}^K$$

$$-\binom{J+L+M}{R+S-K-P-N}\binom{R+S-K}{R-K} I_{DC}^N I_{AB}^K$$

$$-\sum_Q \binom{J+L+M}{R+S-P-K+Q}\binom{R+S-K+Q}{R}\binom{K}{Q} I_{CD}^N I_{BA}^K$$

$$+\sum_Q \binom{J+L+M}{R+S-P-K-N+Q}\binom{R+S-K+Q}{R}\binom{K}{Q} I_{DC}^N I_{BA}^K\Bigg]$$

$$\times\int_\Omega D_R E_B^J(h)\Big(D_S E_A^P E_C^L(f) D_{J+M+L+P+N+K-R-S} E_D^M(g) - (F \leftrightarrow G)\Big).$$

Adding the terms with cyclic permutations, we group terms like

$$D_S E_A^P E_C^L(f)\Big(D_R E_B^J(h) D_{J+M+L+P+N+K-R-S} E_D^M(g) - (H \leftrightarrow G)\Big),$$

and, according to Lemma 2.8 substitute

$$D_S E_A^P E_C^L(f) = D_S \sum (-1)^{|T|}\binom{L+T}{L} E_A^{P-T}\frac{\partial f}{\partial \phi_C^{(L+T)}}$$

$$= \sum_{T,U}(-1)^{|U|+|P|}\binom{U}{P-T}\binom{L+T}{L} D_{S+U-P+T}\frac{\partial^2 f}{\partial \phi_A^{(U)}\partial \phi_C^{(L+T)}}.$$

Then make a change $T \to T-L$

$$\frac{1}{4}\sum(-1)^{|U|+|P|}\binom{U}{L+P-T}\binom{T}{L}[\cdots] D_{S+U+T-L-P}\frac{\partial^2 f}{\partial \phi_A^{(U)}\partial \phi_C^{(T)}}$$

$$\times\Big(D_R E_B^J(h) D_{J+M+L+P+N+K-R-S} E_D^M(g) - (H \leftrightarrow G)\Big),$$

and $S \to S+L+P+N+K$

$$\frac{1}{4}\sum(-1)^{|U|+|P|}\binom{T}{L}\binom{U}{L+P-T}$$

$$\times\Bigg[\binom{J+L+M}{R+S+L+N}\binom{R+S+L+P+N}{R-K} I_{CD}^N I_{AB}^K$$

$$-\binom{J+L+M}{R+S+L}\binom{R+S+L+P+N}{R-K} I_{DC}^N I_{AB}^K$$

$$-\sum_Q \binom{J+L+M}{R+Q+S+L+N}\binom{R+S+Q+P+L+N}{R}\binom{K}{Q} I_{CD}^N I_{BA}^K$$

$$+\sum_{Q}\binom{J+L+M}{R+Q+S+L}\binom{R+Q+S+L+P+N}{R}\binom{K}{Q}I^{N}_{DC}I^{K}_{BA}\Big]$$

$$\times D_{S+U+T+N+K}\frac{\partial^2 f}{\partial\phi^{(U)}_A\partial\phi^{(T)}_C}\Big(D_R E^J_B(h)D_{J+M-R-S}E^M_D(g)-(H\leftrightarrow G)\Big).$$

So, we are able to estimate sums over P

$$\sum_P(-1)^{|P|}\binom{U}{L+P-T}\binom{R+S+L+P+N}{R-K}$$

$$=(-1)^{|L|+|U|+|T|}\binom{T+R+S+N}{R-K-U},$$

$$\sum_P(-1)^{|P|}\binom{U}{L+P-T}\binom{R+S+L+P+N+Q}{R}$$

$$=(-1)^{|L|+|U|+|T|}\binom{T+R+S+N+Q}{R-U},$$

and obtain

$$\frac{1}{4}\sum(-1)^{|L|+|T|}\binom{T}{L}\Big[\binom{J+L+M}{R+S+L+N}\binom{T+R+S+N}{R-K-U}I^{N}_{CD}I^{K}_{AB}$$

$$-\binom{J+L+M}{R+S+L}\binom{T+R+S+N}{R-K-U}I^{N}_{DC}I^{K}_{AB}$$

$$-\sum_Q\binom{J+L+M}{R+Q+S+L+N}\binom{T+R+S+Q+N}{R-U}\binom{K}{Q}I^{N}_{CD}I^{K}_{BA}$$

$$+\sum_Q\binom{J+L+M}{R+Q+S+L}\binom{T+R+Q+S+N}{R-U}\binom{K}{Q}I^{N}_{DC}I^{K}_{BA}\Big]$$

$$\times D_{S+U+T+N+K}\frac{\partial^2 f}{\partial\phi^{(U)}_A\partial\phi^{(T)}_C}\Big(D_R E^J_B(h)D_{J+M-R-S}E^M_D(g)-(H\leftrightarrow G)\Big).$$

Summing over L

$$\sum_L(-1)^{|L|}\binom{T}{L}\binom{L+J+M}{L+R+S+N}=(-1)^{|T|}\binom{J+M}{R+S+N+T},$$

$$\sum_L(-1)^{|L|}\binom{T}{L}\binom{L+J+M}{L+R+S}=(-1)^{|T|}\binom{J+M}{R+S+T},$$

$$\sum_L(-1)^{|L|}\binom{T}{L}\binom{L+J+M}{L+R+S+N+Q}=(-1)^{|T|}\binom{J+M}{R+S+T+N+Q},$$

$$\sum_L(-1)^{|L|}\binom{T}{L}\binom{L+J+M}{L+R+S+Q}=(-1)^{|T|}\binom{J+M}{R+S+T+Q},$$

we get

$$
\begin{aligned}
&\frac{1}{4}\sum\Bigg[\binom{J+M}{R+S+N+T}\binom{T+R+S+N}{R-K-U}I^N_{CD}I^K_{AB}\\
&\qquad -\binom{J+M}{R+S+T}\binom{T+R+S+N}{R-K-U}I^N_{DC}I^K_{AB}\\
&\qquad -\sum_Q\binom{J+M}{R+Q+S+T+N}\binom{T+R+S+Q+N}{R-U}\binom{K}{Q}I^N_{CD}I^K_{BA}\\
&\qquad +\sum_Q\binom{J+M}{R+Q+S+T}\binom{T+R+Q+S+N}{R-U}\binom{K}{Q}I^N_{DC}I^K_{BA}\Bigg]\\
&\times D_{S+U+T+N+K}\frac{\partial^2 f}{\partial\phi^{(U)}_A\partial\phi^{(T)}_C}\Bigg(D_R E^J_B(h)D_{J+M-R-S}E^M_D(g)-(H\leftrightarrow G)\Bigg). \qquad (6.5)
\end{aligned}
$$

Let us make change of indices $S\to -S-R+J+M$. We can sum over Q in the third term of the square brackets

$$
\sum_Q\binom{J+M}{J+M-S+N+Q+T}\binom{J+M-S+N+Q+T}{R-U}\binom{K}{Q}
$$

$$
=\binom{J+M}{R-U}\binom{J+M+K+U-R}{S-N-T}.
$$

Then after interchanging $R\leftrightarrow S$, $J\leftrightarrow M$, $B\leftrightarrow D$, $A\leftrightarrow C$, $N\leftrightarrow K$ and $U\leftrightarrow T$ we see that the first term in the square brackets stands as itself, the new second term is equal to the old third and vice versa. The fourth term transforms into itself[3]:

$$
\sum_Q\binom{J+M}{J+M-S+Q+T}\binom{J+M-S+Q+T+N}{R-U}\binom{K}{Q}
$$

$$
=\sum_Q\binom{J+M}{J+M-R+Q+U}\binom{J+M-R+Q+U+K}{S-T}\binom{N}{Q}.
$$

Evidently, the round bracket in Eq.(6.5) changes its sign, so the expression is zero and the proof is completed.

7 Conclusion

It is clear that the above results can be applied to the field theory on manifolds with a boundary simply by postulating the Rule 4.2 and independently of any reasoning about characteristic functions. The new Poisson structure permits to consider dynamical problems in which boundary values of hamiltonian variables are treated on equal footing with their internal values. The dynamics of field variables on the boundary is determined by both volume and surface parts of the hamiltonian. Any choice of boundary conditions is in fact a

[3]We are able only to verify this fact by computer simulation.

constraint in the phase space and should be treated along with standard procedure of searching for secondary and higher constraints. These boundary conditions do not interfere with the dynamical equations inside the domain until we start solving elliptic type constraints, such as the Gauss law in gauge theories. Then the nonlocal dependence appears, including the dependence of surface variables, and surface part of the hamiltonian begins to influence the equations of internal variables ("divergencies cease to be divergencies" in terminology of Arnowitt, Deser and Misner [23, p.434]).

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Appendix

Here we list different forms in which the new Poisson brackets can be written for the local case (4.2):

1) through the full (*not standard*) variational derivatives defined by the formula (5.2) and with account for Rule 5.4:

$$\{F, G\} = \sum \int \int \frac{\delta F}{\delta \phi_A(x)} \{\phi_A(x), \phi_B(y)\} \frac{\delta G}{\delta \phi_B(y)},$$

2) through higher eulerian operators (2.2):

$$\frac{1}{2} \sum \int_\Omega D_{P+Q} \Big(E_A^P(f) \hat{I}_{AB} E_B^Q(g) - E_A^P(g) \hat{I}_{AB} E_B^Q(f) \Big),$$

where

$$\hat{I}_{AB} = \sum_N I_{AB}^N D_N,$$

3) through Fréchet derivatives (2.3):

$$\{F, G\} = \frac{1}{2} \sum \int_\Omega Tr(D_{f_A} \hat{I}_{AB} D_{g_B} - D_{g_A} \hat{I}_{AB} D_{f_B}),$$

4) through some matrix notations :

$$\{F, G\} = \frac{1}{2} \sum \int_\Omega \Big(\langle \nabla f \cdot C \nabla g \rangle - \langle \nabla g \cdot C \nabla f \rangle \Big),$$

defined below

$$\nabla f = D_L \frac{\partial f}{\partial \phi_A^{(J)}}, \qquad \nabla g = D_M \frac{\partial g}{\partial \phi_B^{(K)}},$$

$$C_{JK,LM,AB} = \binom{J}{L} \binom{K}{M} D_{J+K-L-M} \hat{I}_{AB}.$$